14.4 Tangent Planes and Linear Approximations

Several very close definition are formulated in this section:

a) Tangent plane to the surface S defined as $z = f(x,y)$
at the point $P(x_0, y_0, z_0)$, $z_0 = f(x_0, y_0)$.

b) Linear approximation to the function f at (x_0, y_0).

c) $f(x,y)$ differentiable at (x_0, y_0).

d) Differential dz for a differentiable function

$$z = f(x,y)$$

It is also called total differential.

Consider the function

$$z = f(x,y) = e^{-x^2} + e^{-4y^2}$$

i) Obtain parametric equations for the curve $\mathcal{C}_1$ of intersection of the surface S represented by f with the plane $y=0$.

ii) Compute the tangent vector $\vec{V}_1$ for $\mathcal{C}_1$ at the point $P(-1, 0, f(-1,0))$.

iii) Obtain parametric equations for the curve $\mathcal{C}_2$ of intersection of the surface S with the plane $x=-1$.

iv) Compute the tangent vector $\vec{V}_2$ for $\mathcal{C}_2$ at the point $P(-1, 0, f(-1,0))$

v) Construct the plane that contains the two vectors: $\hat{V}_1$ and $\hat{V}_2$.

i) The Curve $\mathcal{C}_1$ (Intersection of S with plane $y=0$)

$$X = x, \quad y = 0, \quad z = f(x,0) = e^{-x^2} + 1.$$

Introducing a parameter t

$$\mathcal{C}_1: \; X(t) = t, \; y(t) = 0, \; z(t) = e^{-t^2} + 1$$

Therefore,
$$\vec{r}_1(t) = \langle t, 0, e^{-t^2} + 1 \rangle$$
$$\vec{r}_1'(t) = \langle 1, 0, -2t e^{-t^2} \rangle$$

ii) Tangent vector $\vec{V}_1$ of $\mathcal{C}_1$ at $(-1, 0, f(-1,0))$.

Notice that $x = -1$ at $t = -1$

Then,
$$\vec{V}_1 = \langle 1, 0, 2e^{-1} \rangle$$
$$\vec{r}_1'(-1)''$$

iii) The curve $\mathcal{C}_2$

$$x = -1, \quad y = y, \quad z = f(-1, y) = e^{-1} + e^{-4y^2}.$$

Then
$$\mathcal{C}_2: \; X(t) = -1, \; y(t) = t, \; z(t) = e^{-1} + e^{-4t^2}$$
$$\vec{r}_2(t) = \langle -1, t, e^{-1} + e^{-4t^2} \rangle$$
$$\vec{r}_2'(t) = \langle 0, 1, -8t e^{-4t^2} \rangle$$

iv) <u>Tangent vector $\vec{V}_2$ of $\mathcal{C}_2$ at $(-1, 0, f(-1,0))$.</u>

<u>$y=0$ at $t=0$</u>

$$\vec{V}_2 = \langle 0, 1, 0 \rangle = r_2'(0)$$

v) <u>Plane containing $\vec{V}_1$ and $\vec{V}_2$.</u>

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2e^{-1} \\ 0 & 1 & 0 \end{vmatrix} = -2e^{-1}\hat{i} + \hat{j} = \langle -2e^{-1}, 0, 1 \rangle$$

Plane with normal $\vec{n}$ passing through $P(-1, 0, f(-1,0))$.

Since $f(-1,0) = e^{-1}+1$

$P(-1, 0, e^{-1}+1)$.

Therefore, the plane is given by

$$-2e^{-1}(x+1) + z - (e^{-1}+1) = 0$$

or $$-2e^{-1}x + z = e^{-1}+1+2e^{-1} = 3e^{-1}+1$$

or $$\boxed{-2e^{-1}x + z = 3e^{-1}+1} \qquad (4.1)$$

Tangent Plane to a Surface S represented by $z = f(x,y)$, $(x,y) \in D$ at $P_0(x_0, y_0, z_0)$

First, give definition 6 (LaTex Document).

Derivation of the equation for Tangent plane.

a) Curve $\mathcal{C}_1$ intersection of plane $y = y_0$ with S.

$$\mathcal{C}_1: \quad \underset{\text{varies}}{x = x}, \quad \underset{\text{fixed}}{y = y_0}, \quad z = f(\underset{\text{varies}}{x}, \underset{\text{fixed}}{y_0})$$

parametric equ. for $\mathcal{C}_1$

$$\boxed{x(x) = x, \quad y(x) = y_0, \quad z(x) = f(x, y_0).}$$

b) Curve $\mathcal{C}_2$ intersection of plane $x = x_0$ with S.

parametric equ. for $\mathcal{C}_2$

$$\mathcal{C}_2: \boxed{x(y) = x_0, \quad y(y) = y, \quad z(y) = f(x_0, y)}$$

These two curves pass through the point $(x_0, y_0, f(x_0, y_0))$

Show MAPLE Example for

$$f(x,y) = e^{-x^2} + e^{-4y^2}.$$

Corresponding Vector equations

$\mathcal{C}_1: \ \vec{r}_1(x) = \langle x, y_0, f(x, y_0) \rangle,$

$\mathcal{C}_2: \ \vec{r}_2(y) = \langle x_0, y, f(x_0, y) \rangle$

<u>Tangent Vector to $\mathcal{C}_1$ at $P_0(x_0, y_0, f(x_0, y_0))$</u>

$$\vec{r}_1{}'(x) = \left\langle 1, 0, \frac{\partial f}{\partial x}(x, y_0) \right\rangle$$

$$\Rightarrow \quad \vec{V}_1 = \vec{r}_1{}'(x_0) = \left\langle 1, 0, \frac{\partial f}{\partial x}(x_0, y_0) \right\rangle$$

<u>Tangent Vector to $\mathcal{C}_2$ at $P_0(x_0, y_0, f(x_0, y_0))$.</u>

$$\vec{r}_2{}'(y) = \left\langle 0, 1, \frac{\partial f}{\partial y}(x_0, y) \right\rangle$$

$$\Rightarrow \quad \vec{V}_2 = \vec{r}_2{}'(y_0) = \left\langle 0, 1, \frac{\partial f}{\partial y}(x_0, y_0) \right\rangle$$

These two tangent vectors lay in a very special plane <u>Tangent Plane.</u>

Equation of Tangent Plane.

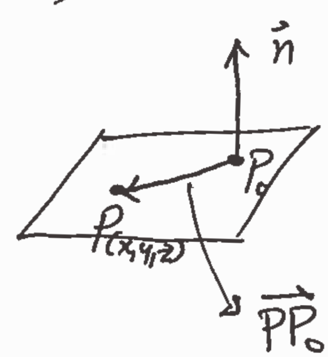
Normal

$$\vec{n} = \vec{V}_1 \times \vec{V}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & \frac{\partial f}{\partial x}(x_0,y_0) \\ 0 & 1 & \frac{\partial f}{\partial y}(x_0,y_0) \end{vmatrix}$$

$$= -\frac{\partial f}{\partial x}(x_0,y_0)\,\hat{\imath} - \frac{\partial f}{\partial y}(x_0,y_0)\,\hat{\jmath} + 1\,\hat{k}$$

Finally, plane containing the two vectors $\vec{V}_1$ and $\vec{V}_2$ and passing through $P_0(x_0, y_0, f(x_0,y_0))$ is given by

$$\vec{PP_0} \cdot \vec{n} = 0$$



$$\left\langle x-x_0,\ y-y_0,\ z-f(x_0,y_0) \right\rangle \cdot \left\langle -\frac{\partial f}{\partial x}(x_0,y_0),\ -\frac{\partial f}{\partial y}(x_0,y_0),\ 1 \right\rangle = 0$$

$$\Rightarrow -\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) - \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0) + z - f(x_0,y_0) = 0$$

$$\Rightarrow \boxed{z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)}$$

0.7 Definition 6: Tangent plane

- Consider a function $f(x, y)$ defined in $\mathcal{D}$ that represents a surface S in $\mathbb{R}^3$.

- This function has continuous first partial derivatives in $\mathcal{D}$.

- Also, $P_0(x_0, y_0, z_0)$ is the center of an open disc $B_r(P_0) \subseteq \mathcal{D}$.

- Let C_1 and C_2 be the two curves obtained by intersecting the surface S with the two planes $x = x_0$ and $y = y_0$, respectively. If T_1 and T_2 are the tangent lines at the point P_0 to the curves C_1 and C_2, respectively,

then the tangent plane to the surface S at P_0 is defined to be the plane that contains both tangent lines T_1 and T_2.

0.8 Theorem 2: Equation of the tangent plane

If $f(x, y)$ and the surface S satisfy the conditions of the previous definition then, an equation of the tangent plane to the surface S, represented by the equation $z = f(x, y)$, at the point P_0 is given by

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

0.9 Definition 7: Linear approximation of f at (x_0, y_0)

For f satisfying the conditions of Definition 1, we define the *linear approximation of f at* (x_0, y_0) as

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

0.10 Definition 8: Differentiability

Assume f has partial derivatives at $(x_0, y_0)) \in \mathcal{D}$, we will say that f is differentiable at (x_0, y_0) if

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

where $\epsilon_1,\ \epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$

0.11 Theorem 3: Sufficient condition for differentiability

If the partial derivatives f_x and f_y exist near (x_0, y_0) and are continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).

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<u>Exercise:</u> # 2 (book 9th ed.)

Find equ. of tangent plane to the graph of Surface: $\boxed{z = f(x,y) = y^2 \sin x.}$ at the point $\boxed{(\pi/2, -2, 4)}$

Tang. plane equ. in general: (at (x_0, y_0, z_0))

$$z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0).$$

In our case,

$\frac{\partial f}{\partial x}(x,y) = y^2 \cos x$, then $\frac{\partial f}{\partial x}(\pi/2,-2) = 4\cos(\pi/2) = 0.$

$\frac{\partial f}{\partial y}(x,y) = 2y \sin x$, then $\frac{\partial f}{\partial y}(\pi/2,-2) = -4\sin(\pi/2) = -4$

Therefore, Tangent Plane is

$$z = 4 + 0(x-\pi/2) - 4(y+2)$$

or $\boxed{z + 4y = -4}$

<u>Remark</u>: Notice that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are both conts at $(\pi/2, -2)$

Linear Approximation

The tangent plane

$$z + 4y = -4$$

is also called a <u>linear approximation to the function $f(x,y)$ at the point</u> $(x_0, y_0) = (\pi/2, -2)$.

$$\boxed{L(x,y) = z = -4y - 4}.$$

The linear approximation at (x_0, y_0) can be used to approximate the value of the function $f(x,y)$ near the point (x_0, y_0).

For example, if we want to approximate

$$f(1.4, -1.8) = (-1.8)^2 \sin(1.4).$$

We evaluate

$$L(1.4, -1.8) = -4(-1.8) - 4 = 3.2.$$

While the actual value is given by

$$f(1.4, -1.8) = (-1.8)^2 \sin(1.4) \simeq 3.19$$